

# Generalized Timelike Mannheim Curves in Minkowski space-time $E_1^4$

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## Abstract

We give a definition of generalized timelike Mannheim curve in Minkowski space-time  $E_1^4$ . The necessary and sufficient conditions for the generalized timelike Mannheim curve obtain. We show some characterizations of generalized Mannheim curve.

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## 1 Introduction

The geometry of curves has long captivated the interests of mathematicians, from the ancient Greeks through to the era of Isaac Newton (1647-1727) and the invention of the calculus. It is branch of geometry that deals with smooth curves in the plane and in the space by methods of differential and integral calculus. The theory of curves is the simpler and narrower in scope because a regular curve in a Euclidean space has no intrinsic geometry. One of the most important tools used to analyze curve is the Frenet frame, a moving frame that provides a coordinate system at each point of curve that is "best adopted" to the curve near that point. Every person of classical differential geometry meets early in his course the subject of Bertrand curves, discovered in 1850 by J. Bertrand. A Bertrand curve is a curve such that its principal normals are the principal normals of a second curve. There are many works related with Bertrand curves in the Euclidean space and Minkowski space, [1]-[3].

Another kind of associated curve is called Mannheim curve and Mannheim partner curve. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: A space curve is a Mannheim curve if and only if its curvature  $\kappa$  and torsion  $\tau$  satisfy the relation

$$k_1 = \beta (k_1^2 + k_2^2)$$

for some constant  $\beta$ . The articles concerning Mannheim curves are rather few. In [4], a remarkable class of Mannheim curves is studied. General Mannheim curves in the Euclidean 3-space are obtained in [5]-[7]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves are given in Euclidean 4-space  $E^4$  by [8]. In this paper, we study the generalized spacelike Mannheim partner curves in 4-dimensional Minkowski space-time. We will give the necessary and sufficient conditions for the generalized spacelike Mannheim partner curves.

## 2 Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in Minkowski space-time  $E_1^4$  are briefly presented in this section. A more complete elementary treatment can be found in [9].

Minkowski space-time  $E_1^4$  is an Euclidean space provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where  $(x_1, x_2, x_3, x_4)$  is a rectangular coordinate system in  $E_1^4$ .

Since  $\langle , \rangle$  is an indefinite metric, recall that a  $\mathbf{v} \in E_1^4$  can have one of the three causal characters; it can be spacelike if  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$  or  $\mathbf{v} = 0$ , timelike if  $\langle \mathbf{v}, \mathbf{v} \rangle < 0$  and null(lightlike) if  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  and  $\mathbf{v} \neq 0$ . Similarly, an arbitrary curve  $\mathbf{c} = \mathbf{c}(t)$  in  $E_1^4$  can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors  $\mathbf{c}'(t)$  are, respectively, spacelike, timelike or null. The norm of  $\mathbf{v} \in E_1^4$  is given by  $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$ . If  $\|\mathbf{c}'(t)\| = \sqrt{|\langle \mathbf{c}'(t), \mathbf{c}'(t) \rangle|} \neq 0$  for all  $t \in I$ , then  $C$  is a regular curve in  $E_1^4$ . A timelike (spacelike) regular curve  $C$  is parameterized by arc-length parameter  $t$  which is given by  $\mathbf{c} : I \rightarrow E_1^4$ , then the tangent vector  $\mathbf{c}'(t)$  along  $C$  has unit length, that is,  $\langle \mathbf{c}(t), \mathbf{c}(t) \rangle = 1$ , ( $\langle \mathbf{c}(t), \mathbf{c}(t) \rangle = -1$ ) for all  $t \in I$ .

Hereafter, curves considered are timelike and regular  $C^\infty$  curves in  $E_1^4$ . Let  $T(t) = \mathbf{c}'(t)$  for all  $t \in I$ , then the vector field  $T(t)$  is timelike and it is called timelike unit tangent vector field on  $C$ .

The timelike curve  $C$  is called special timelike Frenet curve if there exist three smooth functions  $k_1, k_2, k_3$  on  $C$  and smooth non-null frame field  $\{T, N, B_1, B_2\}$  along the curve  $C$ . Also, the functions  $k_1, k_2$  and  $k_3$  are called the first, the second and the third curvature function on  $C$ , respectively. For the  $C^\infty$  special timelike Frenet curve  $C$ , the following Frenet formula is

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

[9].

Here, due to characters of Frenet vectors of the timelike curve,  $T, N, B_1$  and  $B_2$  are mutually orthogonal vector fields satisfying equations

$$\langle T, T \rangle = -1, \quad \langle N, N \rangle = \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 1.$$

For  $t \in I$ , the non-null frame field  $\{T, N, B_1, B_2\}$  and curvature functions  $k_1, k_2$  and  $k_3$  are determined as follows

$$\begin{aligned} 1^{st} \text{ step } T(t) &= \mathbf{c}'(t) \\ 2^{nd} \text{ step } k_1(t) &= \|T'(t)\| > 0 \\ N(t) &= \frac{1}{k_1(t)} T'(t) \\ 3^{rd} \text{ step } k_2(t) &= \|N'(t) - k_1(t) T(t)\| > 0 \\ B_1(t) &= \frac{1}{k_2(t)} (N'(t) - k_1(t) T(t)) \\ 4^{th} \text{ step } B_2(t) &= \varepsilon \frac{1}{\|B_1'(t) + k_2(t) N(t)\|} (B_1'(t) + k_2(t) N(t)) \end{aligned}$$

where  $\varepsilon$  is determined by the fact that orthonormal frame field  $\{T(t), N(t), B_1(t), B_2(t)\}$ , is of positive orientation. The function  $k_3$  is determined by

$$k_3(t) = \langle B_1'(t), B_2(t) \rangle \neq 0.$$

So the function  $k_3$  never vanishes.

In order to make sure that the curve  $C$  is a special timelike Frenet curve, above steps must be checked, from 1<sup>st</sup> step to 4<sup>th</sup> step, for  $t \in I$ .

Let  $\{T, N, B_1, B_2\}$  be the moving Frenet frame along a unit speed timelike curve  $C$  in  $E_1^4$ , consisting of the tangent, the principal normal, the first binormal and the second binormal vector field, respectively. Since  $C$  is a timelike curve, its Frenet frame contains only non-null vector fields.

### 3 Generalized timelike Mannheim curves in $E_1^4$

Mannheim curves are generalized in by [8]. In this paper, we have investigated generalization of timelike Mannheim curves in Minkowski space  $E_1^4$ .

**Definition 3.1** *A special timelike curve  $C$  in  $E_1^4$  is a generalized timelike Mannheim curve if there exists a special timelike Frenet curve  $C^*$  in  $E_1^4$  such that the first normal line at each point of  $C$  is included in the plane generated by the second normal line and the third normal line of  $C^*$  at the corresponding point under  $\phi$ . Here  $\phi$  is a bijection from  $C$  to  $C^*$ . The curve  $C^*$  is called the generalized timelike Mannheim mate curve of  $C$ .*

By the definition, a generalized Mannheim mate curve  $C^*$  is given by the map  $\mathbf{c}^* : I^* \rightarrow E_1^4$  such that

$$\mathbf{c}^*(t) = \mathbf{c}(t) + \beta(t) N(t), \quad t \in I. \quad (3.1)$$

Here  $\beta$  is a smooth function on  $I$ . Generally, the parameter  $t$  isn't an arc-length of  $C^*$ . Let  $t^*$  be the arc-length of  $C^*$  defined by

$$t^* = \int_0^t \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| dt.$$

If a smooth function  $f : I \rightarrow I^*$  is given by  $f(t) = t^*$ , then for  $\forall t \in I$ , we have

$$f'(t) = \frac{dt^*}{dt} = \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| = \sqrt{|-(1 + \beta(t) k_1(t))^2 + (\beta'(t))^2 + (\beta(t) k_2(t))^2|}.$$

The representation of timelike curve  $C^*$  with arc-length parameter  $t^*$  is

$$\begin{aligned} \mathbf{c}^* : I^* &\rightarrow E_1^4 \\ t^* &\rightarrow \mathbf{c}^*(t^*). \end{aligned}$$

For a bijection  $\phi : C \rightarrow C^*$  defined by  $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$ , the reparameterization of  $C^*$  is

$$\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta(t) N(t)$$

where  $\beta$  is a smooth function on  $I$ . Thus, we have

$$\frac{d\mathbf{c}^*(f(t))}{dt} = \frac{d\mathbf{c}^*(t^*)}{dt} \Big|_{t^*=f(t)} f'(t) = f'(t) T^*(f(t)), \quad t \in I.$$

**Theorem 3.1** *If a special timelike Frenet curve  $C$  in  $E_1^4$  is a generalized timelike Mannheim curve, then the following relation between the first curvature function  $k_1$  and the second curvature function  $k_2$  holds:*

$$k_1(t) = -\beta(k_1^2(t) - k_2^2(t)) \quad , \quad t \in I \quad (3.2)$$

where  $\beta$  is a constant number.

**Proof** Let  $C$  be a generalized timelike Mannheim curve and  $C^*$  be the generalized timelike Mannheim mate curve of  $C$ , as following diagram

$$\begin{array}{ccc} & \mathbf{c} & \mathbf{c}^* \\ f : & \overset{\cdot}{I} & \rightarrow \overset{\cdot}{I}^* \\ & \downarrow & \downarrow \\ \phi : & E_1^4 & \rightarrow E_1^4 \end{array}$$

A smooth function  $h$  is defined by  $f(t) = \int \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| dt = t^*$  and  $t^*$  is the arc-length parameter of  $C^*$ . Also  $\phi$  is a bijection which is defined by  $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$ . Thus, the timelike curve  $C^*$  is reparametrized as follows

$$\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta(t) N(t) \quad (3.3)$$

where  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function. By differentiating both sides of equation (3.3) with respect to  $t$ , we have

$$f'(t) T^*(f(t)) = (1 + \beta(t) k_1(t)) T + \beta'(t) N(t) + \beta(t) k_2(t) B_1(t). \quad (3.4)$$

On the other hand, since the first normal line at the each point of  $C$  is lying in the plane generated by the second normal line and the third normal line of  $C^*$  at the corresponding points under bijection  $\phi$ , the vector field  $N(t)$  is given by

$$N(t) = g(t) B_1^*(f(t)) + h(t) B_2^*(f(t))$$

where  $g$  and  $h$  are some smooth functions on  $I \subset \mathbb{R}$ . If we take into consideration

$$\langle T^*(f(t)), g(t) B_1^*(f(t)) + h(t) B_2^*(f(t)) \rangle = 0$$

and the equation (3.4), then we have  $\beta'(t) = 0$ . So we rewrite the equation (3.4) as

$$f'(t) T^*(f(t)) = (1 + \beta k_1(t)) T(t) + \beta k_2(t) B_1(t), \quad (3.5)$$

that is,

$$T^*(f(t)) = \frac{(1 + \beta k_1(t))}{f'(t)} T(t) + \frac{\beta k_2(t)}{f'(t)} B_1(t)$$

where

$$f'(t) = \sqrt{|-(1 + \beta k_1(t))^2 + (\beta k_2(t))^2|}.$$

By taking differentiation both sides of the equations (3.5) with respect to  $t \in I$ , we get

$$\begin{aligned} f'(t) k_1^*(f(t)) N^*(f(t)) &= \left( \frac{1 + \beta k_1(t)}{f'(t)} \right)' T(t) \\ &+ \left( \frac{(1 + \beta k_1(t)) k_1(t) - \beta (k_2(t))^2}{f'(t)} \right) N(t) \\ &+ \left( \frac{\beta k_2(t)}{f'(t)} \right)' B_1(t) + \left( \frac{\beta k_2(t) k_3(t)}{f'(t)} \right) B_2(t). \end{aligned} \quad (3.6)$$

Since

$$\langle N^*(f(t)), g(t) B_1^*(f(t)) + h(t) B_2^*(f(t)) \rangle = 0.$$

The coefficient of  $N(t)$  in equation (3.6) vanishes, that is,

$$(1 + \beta k_1(t)) k_1(t) - \beta (k_2(t))^2 = 0.$$

Thus, this completes the proof.

**Theorem 3.2** In  $E_1^4$ , let  $C$  be a special timelike Frenet curve such that its non-constant first and second curvature functions satisfy the equality  $k_1(s) = -\beta(k_1^2(t) - k_2^2(t))$  for all  $t \in I \subset \mathbb{R}$ . If the timelike curve  $C^*$  given by

$$\mathbf{c}^*(t) = \mathbf{c}(t) + \beta N(t)$$

is a special timelike Frenet curve, then  $C^*$  is a generalized timelike Mannheim mate curve of  $C$ .

**Proof** The arc-length parameter of  $C^*$  is given by

$$t^* = \int_0^t \left\| \frac{d\mathbf{c}^*(t)}{dt} \right\| dt, \quad t \in I.$$

Under the assumption of

$$k_1(t) = -\beta(k_1^2(t) - k_2^2(t)),$$

we obtain  $f'(t) = \sqrt{|1 + \beta k_1(t)|}$ ,  $t \in I$ .

Differentiating the equation  $\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta N(t)$  with respect to  $t$  the we reach

$$f'(t) T^*(f(t)) = (1 + \beta k_1(t)) T(t) + \beta k_2(t) B_1(t).$$

Thus, it is seen that

$$T^*(f(t)) = \left( \frac{1 + \beta k_1(t)}{\sqrt{|1 + \beta k_1(t)|}} T(t) + \frac{\beta k_2(t)}{\sqrt{|1 + \beta k_1(t)|}} B_1(t) \right), \quad t \in I. \quad (3.7)$$

The differentiation of the last equation with respect to  $t$  is

$$\begin{aligned} f'(t) k_1^*(f(t)) N^*(f(t)) &= \left( \sqrt{|1 + \beta k_1(t)|} \right)' T(t) \\ &+ \left( \frac{(1 + \beta k_1(t)) k_1(t) - \beta k_2^2(t)}{\sqrt{|1 + \beta k_1(t)|}} \right) N(t) \\ &+ \left( \frac{\beta k_2(t)}{\sqrt{|1 + \beta k_1(t)|}} \right)' B_1(t) + \left( \frac{\beta k_2(t) k_3(t)}{\sqrt{|1 + \beta k_1(t)|}} \right) B_2(t). \end{aligned} \quad (3.8)$$

From our assumption, we have

$$\frac{k_1(t) + \beta k_1^2(t) - \beta k_2^2(t)}{\sqrt{|1 + \beta k_1(t)|}} = 0.$$

Thus, the coefficient of  $N(t)$  in the equation (3.8) is zero. It is seen from the equation (3.7),  $T^*(f(t))$  is a linear combination of  $T(t)$  and  $B_1(t)$ . Additionally, from equation (3.8),  $N^*(f(t))$  is given by linear combination of  $T(t)$ ,  $B_1(t)$  and  $B_2(t)$ . On the otherhand,  $C^*$  is a special timelike Frenet curve that the vector  $N(t)$  is given by linear combination of  $T^*(f(t))$  and  $N^*(f(t))$ .

Therefore, the first normal line  $C$  lies in the plane generated by the second normal line and third normal line of  $C^*$  at the corresponding points under a bijection  $\phi$  which is defined by  $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$ .

This, completes the proof.

**Remark 3.1** In 4-diemsional Minkowski space  $E_1^4$ , a special timelike Frenet curve  $C$  with curvature functions  $k_1$  and  $k_2$  satisfying  $k_1(t) = -\beta(k_1^2(t) - k_2^2(t))$ , it is not clear that a smooth timelike curve  $C^*$  given by (3.1) is a special Frenet curve. Thus, it is unknown whether the reverse of Theorem 3.1 is true or false.

**Theorem 3.3** *Let  $C$  be a special timelike curve in  $E_1^4$  with non-zero third curvature function  $k_3$ . If there exists a timelike special Frenet curve  $C^*$  in  $E_1^4$  such that the first normal line of  $C$  is linearly dependent with the third normal line of  $C^*$  at the corresponding points  $c(t)$  and  $c^*(t)$ , respectively, under a bijection  $\phi : C \rightarrow C^*$ , iff the curvatures  $k_1$  and  $k_2$  of  $C$  are constant functions.*

**Proof** Let  $C$  be a timelike Frenet curve in  $E_1^4$  with the Frenet frame field  $\{T, N, B_1, B_2\}$  and curvature functions  $k_1, k_2$  and  $k_3$ . Also, we assume that  $C^*$  be a timelike special Frenet curve in  $E_1^4$  with the Frenet frame field  $\{T^*, N^*, B_1^*, B_2^*\}$  and curvature functions  $k_1^*, k_2^*$  and  $k_3^*$ . Let the first normal line of  $C$  be linearly dependent with the third normal line of  $C^*$  at the corresponding points  $C$  and  $C^*$ , respectively. Then the parameterization of  $C^*$  is

$$\mathbf{c}^*(f(t)) = \mathbf{c}(t) + \beta(t)N(t), \quad t \in I. \quad (3.9)$$

If the arc-length parameter of  $C^*$  is given  $t^*$ , then

$$t^* = \int_0^t \sqrt{|-(1 + \beta(t)k_1(t))^2 + (\beta'(t) + (\beta(t)k_2(t))^2|} dt \quad (3.10)$$

and

$$\begin{aligned} f : I &\rightarrow I^* \\ t &\rightarrow f(t) = t^*. \end{aligned}$$

Moreover,  $\phi : C \rightarrow C^*$  is a bijection given by  $\phi(\mathbf{c}(t)) = \mathbf{c}^*(f(t))$ .

Differentiating the equation (3.9) with respect to  $t$  and using Frenet formulas, we get

$$\begin{aligned} f'(t)T^*(f(t)) &= (1 + \beta(t)k_1(t))T(t) + \beta'(t)N(t) \\ &\quad + \beta(t)k_2(t)B_1(t). \end{aligned} \quad (3.11)$$

Since  $B_2^*(f(t)) = \mp N(t)$ , then

$$\langle f'(t)T^*(f(t)), B_2^*(f(t)) \rangle = \left\langle \begin{matrix} (1 + \beta(t)k_1(t))T(t) + \beta'(t)N(t) \\ + \beta(t)k_2(t)B_1(t), \mp N(t) \end{matrix} \right\rangle,$$

that is,

$$0 = \mp \beta'(t).$$

From last equation, it is easily seen that  $\beta$  is a constant. Hereafter, we can denote  $\beta(t) = \beta$ , for all  $t \in I$ .

From the equation (3.10), we have

$$f'(t) = \sqrt{|-(1 + \beta k_1(t))^2 + (\beta k_2(t))^2|} > 0.$$

Thus, we rewrite the equation (3.11) as follows;

$$T^*(f(t)) = \left( \frac{1 + \beta k_1(t)}{f'(t)} \right) T(t) + \left( \frac{\beta k_2(t)}{f'(t)} \right) B_1(t).$$

The differentiation of the last equation with respect to  $t$  is

$$\begin{aligned} f'(t) k_1^*(f(t)) N^*(f(t)) &= \left( \frac{1+\beta k_1(t)}{f'(t)} \right)' T(t) \\ &+ \left( \frac{(1+\beta k_1(t))k_1(t) - \beta k_2^2(t)}{f'(t)} \right) N(t) \\ &+ \left( \frac{\beta k_2(t)}{f'(t)} \right)' B_1(t) + \left( \frac{\beta k_2(t)k_3(t)}{f'(t)} \right) B_2(t). \end{aligned} \quad (3.12)$$

Since  $\langle f'(t) k_1^*(f(t)) N^*(f(t)), B_2^*(f(t)) \rangle = 0$  and  $B_2^*(f(t)) = \mp N(t)$  for all  $t \in I$ , we obtain

$$k_1(t) + \beta k_1^2(t) - \beta k_2^2(t) = 0$$

is satisfied. Then

$$\beta = -\frac{k_1(t)}{k_1^2(t) - k_2^2(t)} \quad (3.13)$$

is a non-zero constant number. Thus, from the equation (3.12), we reach

$$\begin{aligned} N^*(f(t)) &= \frac{1}{f'(t)K(t)} \left( \frac{1+\beta k_1(t)}{f'(t)} \right)' T(t) + \frac{1}{f'(t)K(t)} \left( \frac{\beta k_2(t)}{f'(t)} \right)' B_1(t) \\ &+ \frac{1}{f'(t)K(t)} \left( \frac{\beta k_2(t)k_3(t)}{f'(t)} \right) B_2(t) \end{aligned}$$

where  $K(t) = k_1^*(f(t))$  for all  $t \in I$ . Differentiating the last equation with respect to  $t$ , then we have

$$\begin{aligned} f'(t) [k_1^*(f(t)) T^*(f(t)) + k_2^*(f(t)) B_1(f(t))] &= \left( \frac{1}{f'(t)K(t)} \left( \frac{1+\beta k_1(t)}{f'(t)} \right)' \right)' T(t) \\ &+ \left( \frac{k_1(t)}{f'(t)K(t)} \left( \frac{1+\beta k_1(t)}{f'(t)} \right)' - \frac{k_2(t)}{f'(t)K(t)} \left( \frac{\beta k_2(t)}{f'(t)} \right)' \right) N(t) \\ &+ \left( \left( \frac{1}{f'(t)K(t)} \left( \frac{\beta k_2(t)}{f'(t)} \right)' \right)' - \frac{k_3(t)}{f'(t)K(t)} \left( \frac{\beta k_2(t)k_3(t)}{f'(t)} \right) \right) B_1(t) \\ &+ \left( \left( \frac{1}{f'(t)K(t)} \left( \frac{\beta k_2(t)k_3(t)}{f'(t)} \right)' \right) + \frac{k_3(t)}{f'(t)K(t)} \left( \frac{\beta k_2(t)}{f'(t)} \right)' \right) B_2(t) \end{aligned}$$

for all  $t \in I$ . Considering

$$\langle f'(t) (k_1^*(f(t)) T^*(f(t)) + k_2^*(f(t)) B_1^*(f(t))), B_2^*(f(t)) \rangle = 0$$

and

$$B_2^*(f(t)) = \mp N(t),$$

then we get

$$k_1(t) \left( \frac{1+\beta k_1(t)}{f'(t)} \right)' - k_2(t) \left( \frac{\beta k_2(t)}{f'(t)} \right)' = 0.$$

Arranging the last equation, we find



$$\beta [k_1(t) k_1'(t) - k_2(t) k_2'(t)] f'(t) - [k_1(t) + \beta k_1^2(t) - \beta k_2^2(t)] f''(t) = 0. \quad (3.14)$$

Moreover, the differentiation of the equation (3.13) with respect to  $t$  is

$$k_1'(t) + 2\beta(k_1(t) k_1'(t) - k_2(t) k_2'(t)) = 0.$$

From the above equation, it is seen that

$$-\frac{k_1'(t)}{2} = \beta(k_1(t) k_1'(t) - k_2(t) k_2'(t)). \quad (3.15)$$

Substituting the equations (3.13) and (3.15) into the equation (3.14), we obtain

$$-\frac{k_1'(t)}{2} = 0.$$

This means that the first curvature function is constant (that is, positive constant). Additionally, from the equation (3.15) it is seen that the second curvature function  $k_2$  is positive constant, too.

Conversely, suppose that  $C$  is a timelike Frenet curve  $E_1^4$  in with the Frenet frame field  $\{T, N, B_1, B_2\}$  and curvature functions  $k_1, k_2$  and  $k_3$ . The first curvature function  $k_1$  and the second curvature function  $k_2$  of  $C$  are of positive constant. Thus,  $\frac{k_1}{k_2^2 - k_1^2}$  is a positive constant number, say  $\beta$ .

The representation of timelike curve  $C^*$  with arc-length parameter  $t$  is

$$\begin{aligned} \mathbf{c}^* : I &\rightarrow E_1^4 \\ t &\rightarrow \mathbf{c}^*(t) = \mathbf{c}(t) + \beta(t) N(t). \end{aligned} \quad (3.16)$$

Let  $t^*$  denote the arc-length parameter of  $C^*$ , we have

$$\begin{aligned} f : I &\rightarrow I^* \\ t &\rightarrow t^* = f(t) = \sqrt{|1 + \beta k_1|} t. \end{aligned}$$

Then, we obtain  $f'(t) = \sqrt{|1 + \beta k_1|}$  and

$$\begin{aligned} f'(t) T^*(f(t)) &= T(t) + \beta N'(t) \\ &= (1 + \beta k_1) T(t) + \beta k_2 B_1(t), \end{aligned}$$

that is

$$T^*(f(t)) = \sqrt{|1 + \beta k_1|} T(t) + \frac{\beta k_2}{\sqrt{|1 + \beta k_1|}} B_1(t). \quad (3.17)$$

By differentiating both sides of the above equality with respect to  $t$  we find

$$f'(t) \left. \frac{dT^*(t^*)}{dt^*} \right|_{t^*=f(t)} = \sqrt{|1 + \beta k_1|} T'(t) + \frac{\beta k_2}{\sqrt{|1 + \beta k_1|}} B_1'(t)$$

$$\begin{aligned}
&= \left[ \frac{k_1(1+\beta k_1)-\beta k_2^2}{\sqrt{|1+\beta k_1|}} \right] N(t) + \left[ \frac{\beta k_2 k_3(t)}{\sqrt{|1+\beta k_1|}} \right] B_2(t) \\
&= \left[ \frac{\beta k_2 k_3(t)}{\sqrt{|1+\beta k_1|}} \right] B_2(t).
\end{aligned}$$

Hence, since  $k_3$  doesn't vanish, we get

$$k_1^*(f(t)) = \left\| \frac{dT^*(t^*)}{dt^*} \right\|_{t^*=f(t)} = \varepsilon \frac{\beta k_2 k_3(t)}{1+\beta k_1} > 0$$

where  $\varepsilon = \text{sign}(k_3)$  denotes the sign of function  $k_3$ . That is,  $\varepsilon$  is  $-1$  or  $+1$ . We can put

$$N^*(t^*) = \frac{1}{k_1^*(t^*)} \frac{dT^*(t^*)}{dt^*}, \quad t \in I.$$

Then, we get

$$N^*(f(t)) = \mp B_2(t).$$

Differentiating of the last equation with respect to  $t$ , we reach

$$f'(t) \frac{dN^*(t^*)}{dt^*} \Big|_{t^*=f(t)} = -\varepsilon \frac{k_3}{\sqrt{|1+\beta k_1|}} B_1(t)$$

and we have

$$f'(t) \frac{dN^*(t^*)}{dt^*} \Big|_{t^*=f(t)} - k_1^*(f(t)) T^*(f(t)) = -\varepsilon \frac{\beta k_2 k_3(t)}{\sqrt{|1+\beta k_1|}} T(t) - \varepsilon \sqrt{|1+\beta k_1|} B_1(t).$$

Since  $\varepsilon k_3(t)$  is positive for  $t \in I$ , we have

$$\begin{aligned}
k_2^*(f(t)) &= \left\| \frac{dN^*(t^*)}{dt^*} \Big|_{t^*=f(t)} - k_1^*(f(t)) T^*(f(t)) \right\| \\
&= \sqrt{\frac{\beta^2 k_2^2 (k_3(t))^2}{1+\beta k_1} + (1+\beta k_1) (k_3(t))^2} \\
&= \sqrt{(k_3(t))^2} = \varepsilon k_3(t) > 0.
\end{aligned}$$

Thus, we can put

$$\begin{aligned}
B_1^*(f(t)) &= \frac{1}{k_2^*(f(t))} \left( \frac{dN^*(t^*)}{dt^*} \Big|_{t^*=f(t)} - k_1^*(f(t)) T^*(f(t)) \right) \\
&= -\frac{\beta k_2}{\sqrt{|1+\beta k_1|}} T(t) - \sqrt{|1+\beta k_1|} B_1(t), \quad t \in I.
\end{aligned}$$

Differentiation of the above with respect to  $t$ , we get

$$f'(t) \frac{dB_1^*(t^*)}{dt^*} \Big|_{t^*=f(t)} = \frac{k_2}{\sqrt{|1+\beta k_1|}} N(t) - k_3(t) \sqrt{|1+\beta k_1|} B_2(t).$$

Since  $f'(t) = \sqrt{|1 + \beta k_1|}$  and  $k_2^*(f(t)) N^*(f(t)) = k_3(t) B_2(t)$ , we have

$$\left. \frac{dB_1^*(t^*)}{dt^*} \right|_{t^*=f(t)} + k_2^*(f(t)) N^*(f(t)) = \frac{k_2}{1 + \beta k_1} N(t).$$

Thus, we obtain  $B_2^*(f(t)) = \delta N(t)$  for  $t \in I$ , where  $\delta = \mp 1$ . We must determine whether  $\delta$  is  $-1$  or  $+1$  under the condition that the frame field  $\{T^*(t), N^*(t), B_1^*(t), B_2^*(t)\}$  is of positive orientation.

We have, by  $\det[T(t), N(t), B_1(t), B_2(t)] = 1$  for  $t \in I$ .

$$\begin{aligned} & \det[T^*(t), N^*(t), B_1^*(t), B_2^*(t)] \\ &= \det \begin{bmatrix} \sqrt{|1 + \beta k_1|} T(t) + \frac{\beta k_2}{\sqrt{|1 + \beta k_1|}} B_1(t), \\ \varepsilon B_2(t), -\frac{\beta k_2}{\sqrt{|1 + \beta k_1|}} T(t) - \sqrt{|1 + \beta k_1|} B_1(t), \delta N(t) \end{bmatrix} \\ &= \varepsilon \delta \left( (1 + \beta k_1) - \frac{\beta^2 k_2^2}{1 + \beta k_1} \right) = \varepsilon \delta \end{aligned}$$

and  $\det[T^*(t), N^*(t), B_1^*(t), B_2^*(t)] = 1$  for any  $t \in I$ . Therefore, we get  $\varepsilon = \delta$ . Thus, we get

$$B_2^*(f(t)) = \varepsilon N(t)$$

and

$$\begin{aligned} k_3^*(f(t)) &= \left\langle \left. \frac{dB_1^*(t^*)}{dt^*} \right|_{t^*=f(t)}, B_2^*(f(t)) \right\rangle \\ &= \varepsilon \frac{k_2}{1 + \beta k_1}, \quad t \in I. \end{aligned}$$

By the above facts,  $C^*$  is a special Frenet curve in  $E_1^4$  and the first normal line at each point of  $C$  is the third normal line of  $C^*$  at corresponding each point under the bijection  $\phi: c \rightarrow \phi(c(t)) = c^*(f(t)) \in C^*$ .

Thus, the proof is completed.

The following theorem gives a parametric representation of a generalized time-like Mannheim curves  $E_1^4$ .

**Theorem 3.4** *Let  $C$  be a timelike special curve defined by*

$$\mathbf{c}(s) = \begin{bmatrix} \beta \int f(s) \cosh s \, ds \\ \beta \int f(s) \sinh s \, ds \\ \beta \int f(s) g(s) \, ds \\ \beta \int f(s) h(s) \, ds \end{bmatrix}, \quad s \in U \subset \mathbb{R}.$$

Here,  $\beta$  is a non-zero constant number,  $g: U \rightarrow \mathbb{R}$  and  $h: U \rightarrow \mathbb{R}$  are any smooth functions and the positive valued smooth function  $f: U \rightarrow \mathbb{R}$  is given by

$$f = (1 - g^2(s) - h^2(s))^{-3/2} \left( 1 - g^2(s) - h^2(s) + \dot{g}^2(s) + \dot{h}^2(s) - (\dot{g}(s)h(s) - g(s)\dot{h}(s))^2 \right)^{-5/2} \left[ - \left( 1 - g^2(s) - h^2(s) + \dot{g}^2(s) + \dot{h}^2(s) - (\dot{g}(s)h(s) - g(s)\dot{h}(s))^2 \right)^3 + (1 - g^2(s) - h^2(s))^3 \left( \begin{aligned} &-(g(s) - \ddot{g}(s))^2 - (h(s) - \ddot{h}(s))^2 \\ &- \left( (g(s)\dot{h}(s) - \dot{g}(s)h(s)) - (\dot{g}(s)\ddot{h}(s) - \ddot{g}(s)\dot{h}(s)) \right)^2 \\ &+ (g(s)\ddot{h}(s) - \ddot{g}(s)h(s))^2 \end{aligned} \right) \right],$$

for  $s \in U$ . Then the curvature functions  $k_1$  and  $k_2$  of  $C$  satisfy

$$k_1 = -\beta (k_1^2 - k_2^2).$$

at the each point  $\mathbf{c}(s)$  of  $C$ .

**Proof** Let  $C$  be a timelike special curve defined by

$$\mathbf{c}(s) = \begin{bmatrix} \beta \int f(s) \cosh s \, ds \\ \beta \int f(s) \sinh s \, ds \\ \beta \int f(s) g(s) \, ds \\ \beta \int f(s) h(s) \, ds \end{bmatrix}, \quad s \in U \subset \mathbb{R}$$

where  $\beta$  is a non-zero constant number,  $g$  and  $h$  are any smooth functions.  $f$  is a positive valued smooth function. Thus, we obtain

$$\dot{\mathbf{c}}(s) = \begin{bmatrix} \beta f(s) \cosh s \\ \beta f(s) \sinh s \\ \beta f(s) g(s) \\ \beta f(s) h(s) \end{bmatrix}, \quad s \in U \subset \mathbb{R} \quad (3.18)$$

where the subscript prime  $(\cdot)$  denotes the differentiation with respect to  $s$ . The arc-length parameter  $t$  of  $C$  is given by

$$t = \psi(s) = \int_{s_0}^s \|\dot{\mathbf{c}}(s)\| \, ds$$

where  $\|\dot{\mathbf{c}}(s)\| = \beta f(s) \sqrt{-1 + g^2(s) + h^2(s)}$ .

If  $\varphi$  denotes the inverse function of  $\psi : U \rightarrow I \subset \mathbb{R}$ , then  $s = \varphi(t)$  and we get

$$\varphi'(t) = \left\| \frac{d\mathbf{c}(s)}{ds} \Big|_{s=\varphi(t)} \right\|^{-1}, \quad t \in I$$

where the prime  $(\cdot)$  denotes the differentiation with respect to  $t$ .

The unit tangent vector  $T(t)$  of the curve  $C$  at the each point  $\mathbf{c}(\varphi(t))$  is given by

$$T(t) = (-1 + g^2(\varphi(t)) + h^2(\varphi(t)))^{-1/2} \begin{bmatrix} \cosh(\varphi(t)) \\ \sinh(\varphi(t)) \\ g(\varphi(t)) \\ h(\varphi(t)) \end{bmatrix}, \quad t \in I. \quad (3.19)$$

Some simplifying assumptions are made for the sake of brevity as follows;

$$\begin{aligned} \sinh &:= \sinh(\varphi(t)) & , & & \cosh &:= \cosh(\varphi(t)) \\ f &:= f(\varphi(t)) & , & & g &:= g(\varphi(t)) & , & & h &:= h(\varphi(t)), \\ \dot{g} &:= \dot{g}(\varphi(t)) = \left. \frac{dg(s)}{ds} \right|_{s=\varphi(t)} & , & & \dot{h} &:= \dot{h}(\varphi(t)) = \left. \frac{dh(s)}{ds} \right|_{s=\varphi(t)}, \\ \ddot{g} &:= \ddot{g}(\varphi(t)) = \left. \frac{d^2g(s)}{ds^2} \right|_{s=\varphi(t)} & , & & \ddot{h} &:= \ddot{h}(\varphi(t)) = \left. \frac{d^2h(s)}{ds^2} \right|_{s=\varphi(t)}, \\ \varphi' &:= \varphi'(t) = \left. \frac{d\varphi}{dt} \right|_t & , & & & & & & & \\ A &:= 1 - g^2 - h^2 & , & & B &:= -g\dot{g} - h\dot{h} & , & & C &:= -\dot{g}^2 - \dot{h}^2, \\ D &:= -g\ddot{g} - h\ddot{h} & , & & E &:= -\dot{g}\ddot{g} - \dot{h}\ddot{h} & , & & F &:= \ddot{g}^2 + \ddot{h}^2. \end{aligned}$$

Thus, we get

$$\dot{A} = 2B \quad , \quad \dot{B} = C + D \quad , \quad \dot{C} = 2E \quad , \quad \varphi' = \beta^{-1} f^{-1} A^{-1/2}.$$

So, we rewrite the equation (3.19) as

$$T := T(t) = A^{-1/2} \begin{bmatrix} \cosh \\ \sinh \\ g \\ h \end{bmatrix}. \quad (3.20)$$

Differentiating the last equation with respect to  $t$ , we find

$$T' = \varphi' \begin{bmatrix} -\frac{1}{2}A^{-3/2}\dot{A}\cosh + A^{-1/2}\sinh \\ -\frac{1}{2}A^{-3/2}\dot{A}\sinh + A^{-1/2}\cosh \\ -\frac{1}{2}A^{-3/2}\dot{A}g + A^{-1/2}\dot{g} \\ -\frac{1}{2}A^{-3/2}\dot{A}h + A^{-1/2}\dot{h} \end{bmatrix},$$

that is,

$$T' = -\varphi' A^{-1/2} \begin{bmatrix} A^{-1}B\cosh - \sinh \\ A^{-1}B\sinh - \cosh \\ A^{-1}Bg - \dot{g} \\ A^{-1}Bh - \dot{h} \end{bmatrix}. \quad (3.21)$$

From the last equation, we find

$$k_1 := k_1(t) = \|T'(t)\| = \varphi' A^{-1} (A - AC + B^2)^{1/2}. \quad (3.22)$$

By the fact that  $N(t) = (k_1(t))^{-1}T'(t)$ , we get

$$N := N(t) = -A^{1/2}(A - AC + B^2)^{-1/2} \begin{bmatrix} A^{-1}B \cosh - \sinh \\ A^{-1}B \sinh - \cosh \\ A^{-1}Bg - \dot{g} \\ A^{-1}Bh - \dot{h} \end{bmatrix}.$$

In order to get second curvature function  $k_2$ , we need to calculate  $k_2(t) = \|N'(t) - k_1(t)T(t)\|$ . After a long process of calculations and using abbreviations, we obtain

$$N' - k_1T = \varphi' A^{-3/2}(A - AC + B^2)^{-3/2} \begin{bmatrix} (P + Q) \cosh - R \sinh \\ (P + Q) \sinh - R \cosh \\ Pg - R\dot{g} + Q\ddot{g} \\ Ph - R\dot{h} + Q\ddot{h} \end{bmatrix} \quad (3.23)$$

where

$$\begin{aligned} P &= (A - AC + B^2)(B^2 - AC - AD) - (A - AC + B^2)^2 \\ &\quad + AB(B - AE + BD), \\ Q &= A^2(A - AC + B^2), \\ R &= A^2(B - AE + BD). \end{aligned} \quad (3.24)$$

If we simplify  $P$  then we have

$$P = A^2(C - BE - D + CD - 1).$$

Therefore, we rewrite the equations (3.23) and (3.24) as

$$N' - k_1T = \varphi' A^{-1/2}(A - AC + B^2)^{-3/2} \begin{bmatrix} (\tilde{P} + \tilde{Q}) \cosh - \tilde{R} \sinh \\ (\tilde{P} + \tilde{Q}) \sinh - \tilde{R} \cosh \\ \tilde{P}g - \tilde{R}\dot{g} + \tilde{Q}\ddot{g} \\ \tilde{P}h - \tilde{R}\dot{h} + \tilde{Q}\ddot{h} \end{bmatrix} \quad (3.25)$$

where

$$\begin{aligned} \tilde{P} &= C - D + CD - BE - 1, \\ \tilde{Q} &= A - AC + B^2, \\ \tilde{R} &= B - AE + BD. \end{aligned} \quad (3.26)$$

Consequently, from the equations (3.25) and (3.26), we have

$$\|N' - k_1T\|^2 = (\varphi')^2 A(A - AC + B^2)^{-3} \begin{bmatrix} -(\tilde{P} + \tilde{Q})^2 + \tilde{R}^2 + \tilde{P}^2(g^2 + h^2) + \tilde{R}^2(\dot{g}^2 + \dot{h}^2) \\ + \tilde{Q}^2(\ddot{g}^2 + \ddot{h}^2) - 2\tilde{P}\tilde{R}(g\dot{g} + h\dot{h}) \\ - 2\tilde{R}\tilde{Q}(\dot{g}\ddot{g} + \dot{h}\ddot{h}) + 2\tilde{P}\tilde{Q}(g\ddot{g} + h\ddot{h}) \end{bmatrix}.$$

Substituting the abbreviations into the last equation, we have

$$\|N' - k_1 T\|^2 = (\varphi')^2 A(A - AC + B^2)^{-3} [-\tilde{P}^2 A - 2\tilde{P}\tilde{Q} - \tilde{Q}^2 + \tilde{R}^2 - \tilde{R}^2 C + \tilde{Q}^2 F + 2\tilde{P}\tilde{R}B + 2\tilde{R}\tilde{Q}E - 2\tilde{P}\tilde{Q}D].$$

After substituting the equation (3.26) into the last equation and simplifying it, we get

$$\begin{aligned} k_2^2 &= \|N' - k_1 T\|^2 \\ &= (\varphi')^2 A(A - AC + B^2)^{-2} [(A - AC + B^2)(1 - F) + (C - 1)(1 + D)^2 - 2BE(1 + D) + AE^2]. \end{aligned}$$

Moreover, from the equation (3.22) it is seen that

$$k_1^2 = (\varphi')^2 A^{-2} (A - AC + B^2).$$

The last two equation gives us

$$\begin{aligned} k_2^2 - k_1^2 &= (\varphi')^2 A^{-2} (A - AC + B^2)^{-2} \left[ -(A - AC + B^2)^3 \right. \\ &\quad \left. + A^3 \left( (A - AC + B^2)(1 - F) + (C - 1)(1 + D)^2 - 2BE(1 + D) + AE^2 \right) \right]. \end{aligned}$$

By the fact  $\varphi' = \beta^{-1} f^{-1} A^{-1/2}$ , we obtain

$$\begin{aligned} k_2^2 - k_1^2 &= \beta^{-2} f^{-2} A^{-3} (A - AC + B^2)^{-2} \left[ (A - AC + B^2)^3 \right. \\ &\quad \left. + A^3 \left( (A - AC + B^2)(1 - F) + (C - 1)(1 + D)^2 - 2BE(1 + D) + AE^2 \right) \right] \end{aligned} \tag{3.27}$$

and

$$k_1 = \beta^{-1} f^{-1} A^{-3/2} (A - AC + B^2)^{1/2}.$$

According to our assumption

$$\begin{aligned} f &= (1 - g^2 - h^2)^{-3/2} \left( 1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 - (\dot{g}h - g\dot{h})^2 \right)^{-5/2} \\ &\quad \left[ - \left( 1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 - (\dot{g}h - g\dot{h})^2 \right)^3 \right. \\ &\quad \left. + (1 - g^2 - h^2)^3 \left( -(g - \ddot{g})^2 - (h - \ddot{h})^2 - ((g\dot{h} - \dot{g}h) - (\dot{g}\ddot{h} - \ddot{g}h))^2 + (g\ddot{h} - \ddot{g}h)^2 \right) \right], \end{aligned}$$

we obtain

$$f = A^{-3/2} (A - AC + B^2)^{-5/2} \left[ \begin{aligned} &(A - AC + B^2)^3 \\ &+ A^3 \left( \begin{aligned} &(A - AC + B^2)(1 - F) + (C - 1)(1 + D)^2 \\ &- 2BE(1 + D) + AE^2 \end{aligned} \right) \end{aligned} \right].$$

Substituting the above equation into the equation into the equations (3.27) and (3.28), we obtain

$$k_1 = -\beta (k_1^2 - k_2^2) .$$

The proof is completed.

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